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S. W. STRATTON, DIRECTOR

No. 152

INVESTIGATION OF THE COMPRESSIVE STRENGTH OF SPRUCE STRUTS OF RECTANGULAR CROSS SECTION AND THE DERIVATION OF FORMULAS SUITABLE FOR USE IN AIRPLANE DESIGN

BY

JAMES E. BOYD, Professor of Mechanics, Ohio State University
Expert on Aviation Problems at Bureau of Standards

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INVESTIGATION OF THE COMPRESSIVE STRENGTH OF SPRUCE STRUTS OF RECTANGULAR CROSS SECTION AND THE DERIVATION OF FORMULAS SUITABLE FOR USE IN AIRPLANE DESIGN

By James E. Boyd

CONTENTS

I. INTRODUCTION

These experiments were undertaken for the purpose of finding suitable formulas and constants for the design of spruce struts for airplanes. Test specimens of rectangular cross section (approximately square) were used, as these sections are easy to prepare and the necessary calculations are easily made. The constants and formulas which have been obtained apply to stream-line or

other sections, provided they are uniform throughout their length. The constants apply to struts of tapering section, and the formulas may also be applied, provided a moment of inertia is used which is somewhat less than that of the maximum section. Theoretical methods of computing the strength of tapered struts are given in the last pages of the paper.

The experimental work was conducted under the general direction of Prof. John H. Nelson, engineer physicist of the Bureau of Standards, who decided upon many of the methods employed. C. P. Hoffman and L. J. Larson assisted in making the measurements. The curves and drawings were finished by Prof. C. L. Svensen, of the Ohio State University. Prof. R. D. Bohannan made some suggestions in connection with the theoretical work.

II. ARRANGEMENT OF COMPRESSION TESTS

1. TEST PIECES

In the preparation of the struts, one edge of the plank was first straightened on a jointer. A strip a little over 1.75 inches square was cut off with a saw and planed to size. Each plank furnished five strips designated as A, B, C, D, and E, in order, and also a narrow strip from which endurance test pieces were taken.

The planed strips were cut to length with a back saw and the ends finished by grinding on an emery wheel mounted in a lathe. The strut was held in position perpendicular to the plane of the emery wheel by a pair of right-angled V guides mounted on the bed. These guides were about 2 feet apart when used with the longer struts, so that the end was perpendicular to the last 2 feet of the strut and not perpendicular to its entire length in those cases in which the strut was not perfectly straight.

2. COMPRESSION MEASUREMENTS

For the two shorter lengths the compression was measured by means of a pair of Berry strain gages. For all other lengths a pair of 30-inch Howard gages were used.

To support the gages, a pair of steel pins, each 0.237 inch in diameter, were driven through holes drilled in the strut at the ends of the gage length, these holes being slightly smaller than the pins to give a driving fit. The struts were tested in a horizontal position, with the steel pins also horizontal.

The conical points of the strain gages rested in small holes drilled in the upper surface of the pin. The holes were 2.75 inches apart, which placed the gage points one-half inch from the

vertical surface of the strut. The strain gages were held firmly in place by means of 3-pound weights suspended from stirrups which were attached to the gages directly above the points.

3. DEFLECTION MEASUREMENTS

The deflection of the two shorter lengths of the square-end struts was measured by means of a pair of Ames dials mounted in a metal yoke which was supported by the base of the testing machine. The vertical deflection of the two shorter lengths of round-end struts was measured by an Ames dial supported by a wooden beam which rested on a pair of clamps attached to the strut near the ends.

For all other lengths a pair of Johnson dials were used.

4. TESTING MACHINES

The two shorter lengths of square-end struts were tested on the 230 000-pound Emery machine, and all the others in the 2 300 000-pound Emery machine.

Both these Emery machines at the Bureau of Standards are of the horizontal type.

5. ADJUSTMENT IN THE MACHINE

One end of the strut was placed squarely against one head of the Emery machine, and a light load applied, after which the adjustable head was rotated until an even bearing was secured at the other end. This position of best bearing was determined by means of a Starrett gage of thin sheet metal inserted between the strut and the head. It was also tested by observation of the load on the strut as given by the reading of the balance beam of the testing machine. The position of the head giving the least load is the one required.

For the longer struts it was found advisable to fix the heads nearly parallel, but slightly wider apart at the top than at the bottom. In the case of a strut which was not initially straight, a light load (200 to 300 pounds) was found sufficient to bring it to full bearing and make it more nearly straight than it was before loading.

6. COUNTERWEIGHTS

Each Howard gage with 30-inch rod weighs 3 pounds. With a 3-pound weight on each end, the total weight of the two gages is 18 pounds. This load caused considerable deflection of the longer struts. Nine-pound weights, fastened to strings passing

over light pulleys, were attached to the strut at each gage point to counterbalance the weight of these instruments. Counterweights were not used with the two short lengths of round-end struts, nor with the three short lengths of square-end struts. The weight of the strut itself was not counterbalanced. Struts with initial bend in the vertical plane were placed in the machine convex upward, so that their weight tended to straighten them. For a few struts with relatively large initial deflection upward the counterweights were reduced.

III. SQUARE-END STRUTS

1. EXPERIMENTAL RESULTS

The struts for the square-end tests were taken from six 12-foot planks. There were 10 lengths, from 12⅝ inches, for which $\frac{L}{r}$ is 25, to 10 feet 6¼ inches, for which $\frac{L}{r}$ is 250. The planks were numbered from 1 to 6. The schedule was so arranged that, with a few exceptions (made necessary in order to reserve one piece of each plank for a bending test), one strut of each length was taken from each plank. The complete schedule of struts is given in Table 3.

Most of the struts of the two shorter lengths failed by compression at the transverse pins which supported the gages. This seemed to indicate that a cylindrical pin with higher modulus of elasticity than the material which surrounds it was a source of weakness in a compression member. To investigate this question two 12⅝-inch struts, C–5 and E–6, were tested with gages supported by clamps instead of pins. Each failed at a lower load than the strut from the same plank which had been tested with the pins. If any conclusion may be drawn from these few tests, it is that the pins do not seriously weaken the strut.

The summary of the data for the eight struts of 12⅝-inch length is given in Table 3.

Table 1 gives the complete data of the test of square-end strut B–4, 4 feet 2½ inches in length, for which $\frac{L}{r}$ is 100. Columns II and V give the compression at the north and south gages, respectively. The load was somewhat eccentric, so that the reading at the north gage is considerably larger than at the south gage. The gages were 2.75 inches apart, while the width of the strut was a little under 1.75 inches. The actual difference of compression at the faces of the strut is seven-elevenths of the difference of the

gage readings. To get column III, subtract two-elevenths of the difference of the readings of columns II and V from the reading of column II. To get column VI, add the same amount to the the readings of column V. Column IV gives the unit compression in the north face of the strut, and column VII gives the same for the south face. Column VIII gives the average unit compression.

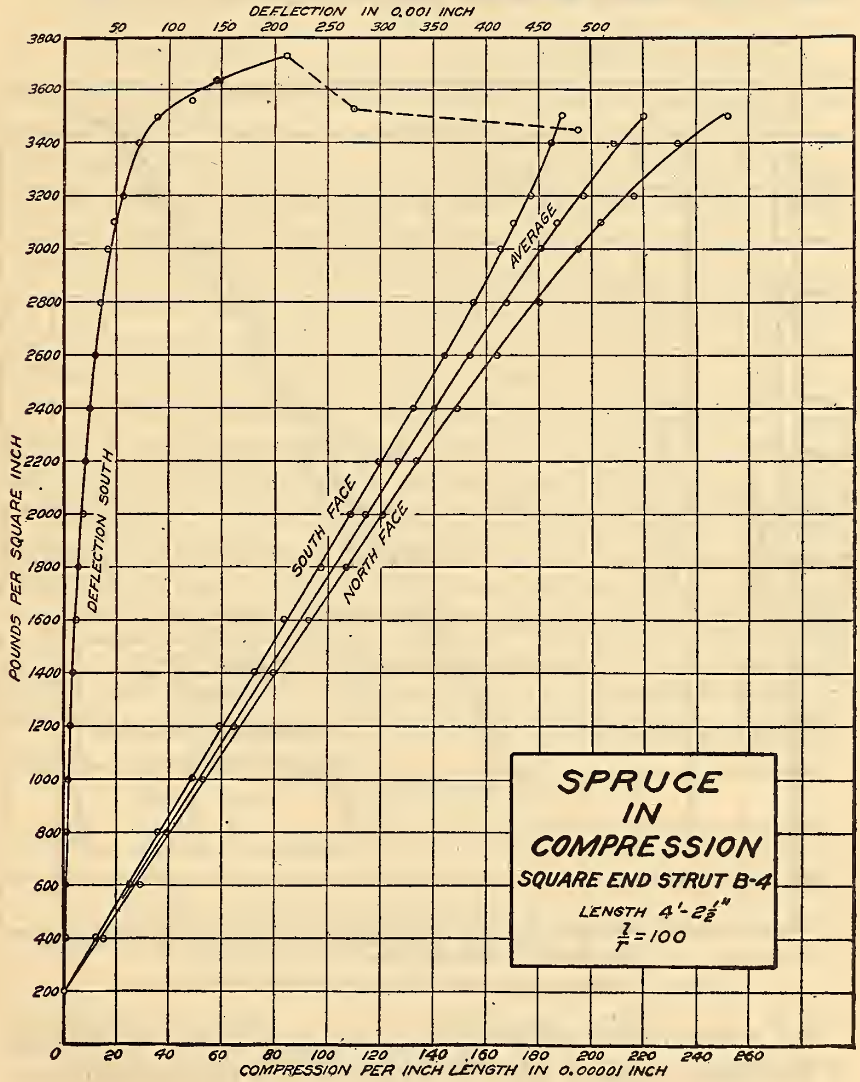

FIG. 1.—*Deflection and compression of square-end spruce strut*

Fig. 1 shows the stress-strain diagrams for this strut, together with the deflection in the horizontal plane. The greater compression is in the north face, which is the concave side when the deflection is south.

Table 2 gives similar data for a second strut, C–4, of length 5 feet 3⅛ inches, for which the slenderness ratio is approximately 125. In this case the horizontal eccentricity was small, so that the readings of the two instruments are nearly equal. The readings of the north gage are the larger until the last load is reached, when that of the south gage becomes the larger. At the same time the horizontal deflection changes from south to north.

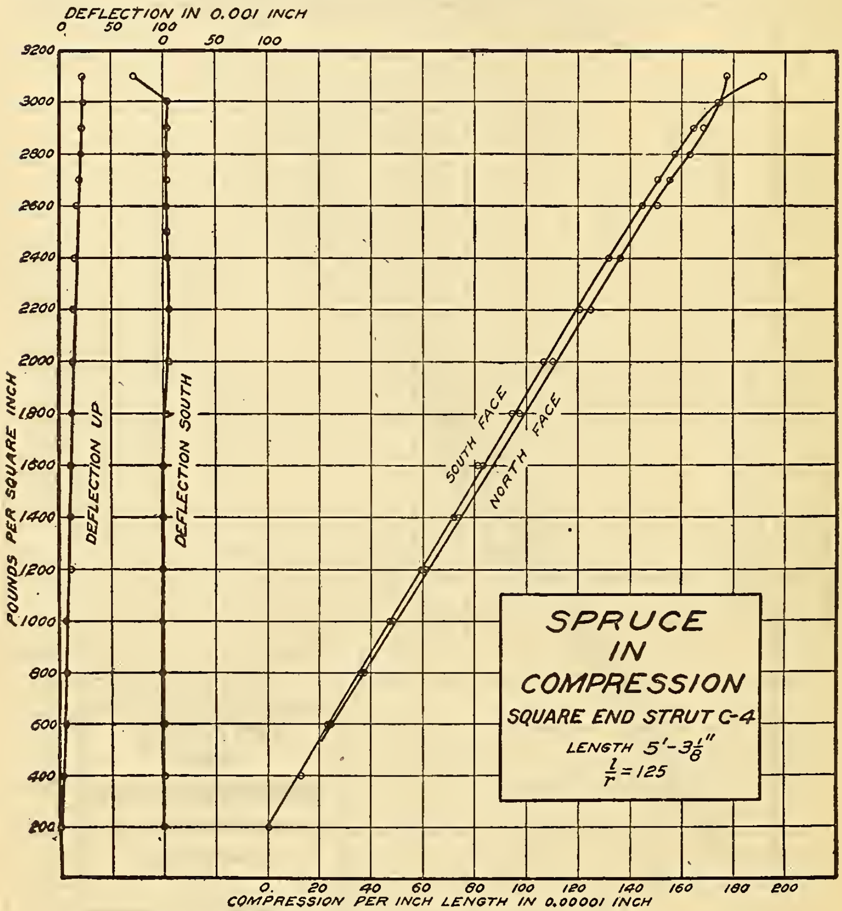

FIG. 2.—*Deflection and compression of square-end spruce strut*

Fig. 2 shows the stress-strain diagrams for the two vertical faces of this strut, together with the deflections in the horizontal and the vertical planes.

Tables 1 and 2 and Figs. 1 and 2 are representative cases of the tests of square-end struts. Generally the longer struts were not carried to compression failure. When the deflection reached about

0.875 inch, so that the center of the section at the middle of the length was in line with the edges of the sections at the ends, the strut rotated on the edges at the ends and came into the condition of a round-end column. The load for this condition was much reduced, and with continued compression the load became still smaller.

Fig. 3 shows the unit load and the average unit compression for all the square-end struts of slenderness ratios 100, 150, and 200. The broken-line extensions of some of the curves are carried up to the ultimate unit loads as found after the removal of the gages which measured the compression. These extensions give the true ultimate load, but not necessarily the real form of the curves. The straight-line part of each curve is extended upward to a convenient point for finding the modulus of elasticity.

Similar curves were drawn for all the struts tested. From these the modulus of elasticity and the proportional elastic limit given in Table 3 were obtained.

There were eight struts of length 3 feet 1⅞ inches. The strut C–6 was cross-grained and broke suddenly at a load which was low for a strut from this plank. Strut A–4 had a curved grain near one of the pins, evidently due to a knot in the log near plank 4 at this point. This strut failed by shear along the curved grain. As these two results were evidently lower than the normal, two additional struts, B–2_a and B–2_b, were cut from the long strut B–2 after it had been tested to its maximum load; and a third strut, C–3, was taken from the 10 foot 6¼ inch strut, C–3. The results from these three struts were used with those from the original five struts in calculating the average ultimate strength for this length.

2. TEST OF FORMULAS

Curve *I* of Fig. 4 is plotted with slenderness ratios as abscissas and the average ultimate strengths of each length as ordinates. The figure also shows the maximum and minimum ultimate strengths of each set. From Table 3 it is seen that all the struts from plank 1, except A–1, gave a low modulus of elasticity and a low ultimate strength. For the slenderness ratio of 150, for instance, E–1 has an ultimate strength of only 1200 pounds per square inch, while the next one has 1993 pounds per square inch. If the results from plank 1 were omitted from Fig. 4, the range from minimum to maximum would be greatly reduced.

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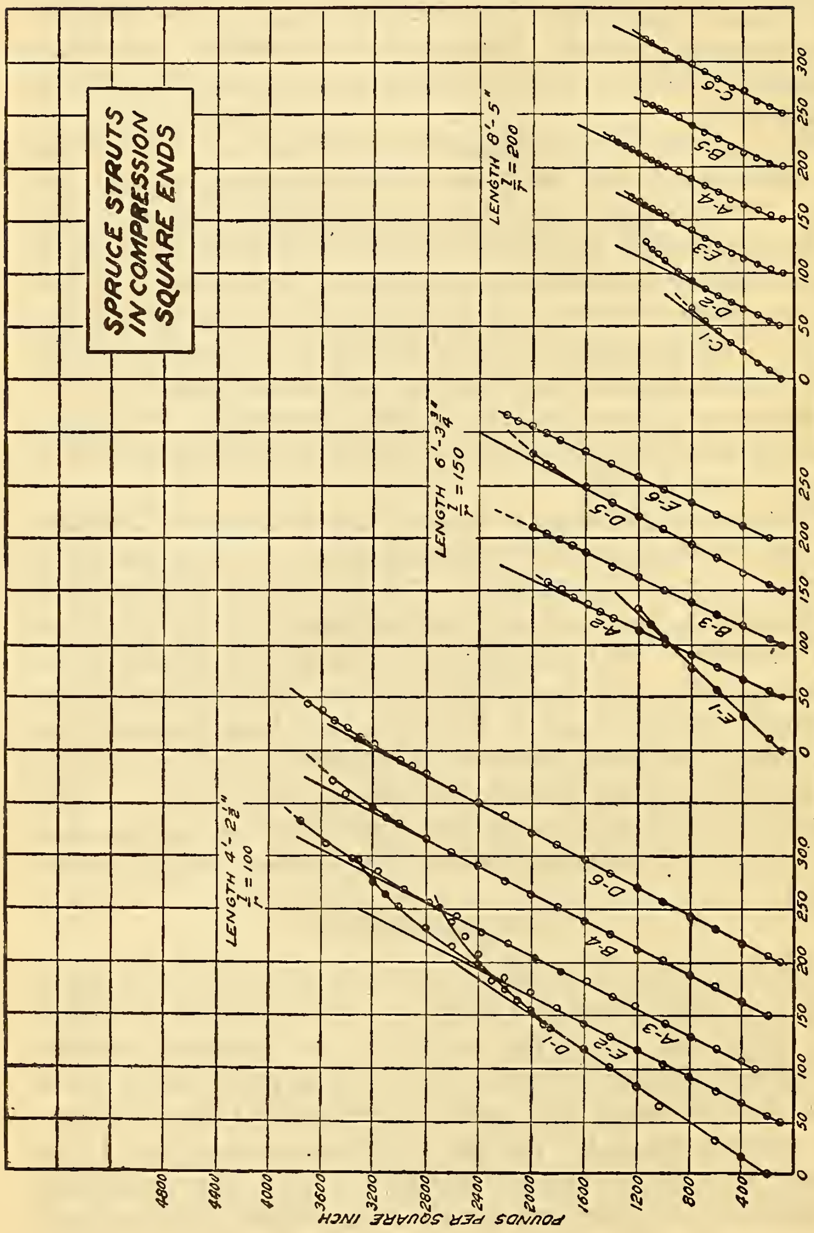

Fig. 3.—Compression curves of three groups of square-end spruce struts

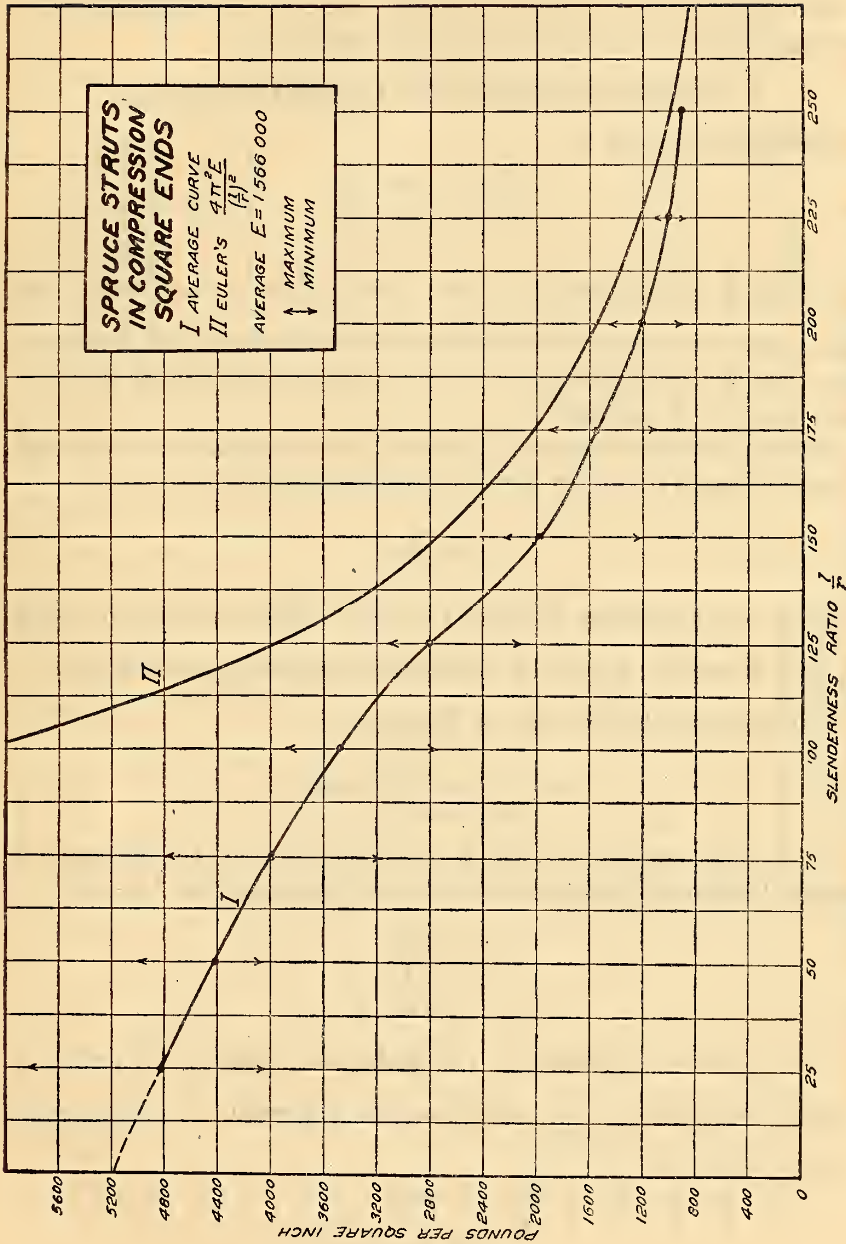

FIG. 4.—*Relation of ultimate strength of square-end spruce struts to slenderness ratio, and comparison with Euler's curve*

The average E from all these square-end struts is 1 566 000 pounds per square inch. Curve *II* of Fig. 4 is plotted from Euler's formula, with this value of the modulus of elasticity, under the assumption that these struts had fixed ends.

3. RANKINE'S FORMULA FOR SQUARE-END STRUTS

Rankine's formula is

$$\frac{P}{A}=\frac{S_u}{1+q\left(\frac{L}{r}\right)^2}$$

in which $\frac{P}{A}$ is the ultimate load in pounds per square inch, S_u is the ultimate compressive strength of the material in the form of a short block, L is the length of the column, r is its radius of gyration, and q is a constant.

Ritter's rational value of q, which makes Rankine's curve approach Euler's curve for large slenderness ratios, is

$$q=\frac{S_u}{\pi^2 E}.$$

From the extension of curve *I* of Fig. 4 back to the zero value of $\frac{L}{r}$, it is evident that S_u is about 5200 pounds per square inch. Using this with the value of E above,

$$q=\frac{1}{2970}=\frac{1}{3000} \text{ nearly.}$$

Using this value of q (calculated from the E of the square-end tests), Rankine's formula for round-end spruce struts becomes

$$\frac{P}{A}=\frac{5200}{1+\frac{L^2}{3000\ r^2}}$$

For square-end struts, if the ends are regarded as perfectly fixed, q becomes $\frac{1}{12\,000\,r^2}$ and Rankine's formula for square-end spruce struts is

$$\frac{P}{A}=\frac{5200}{1+\frac{L^2}{12\,000\,r^2}}.$$

Fig. 5 shows Rankine's curve plotted from this last formula, the small circles of the figure being the average results of the test (used in plotting curve *I* of Fig. 4). It is evident that the formula

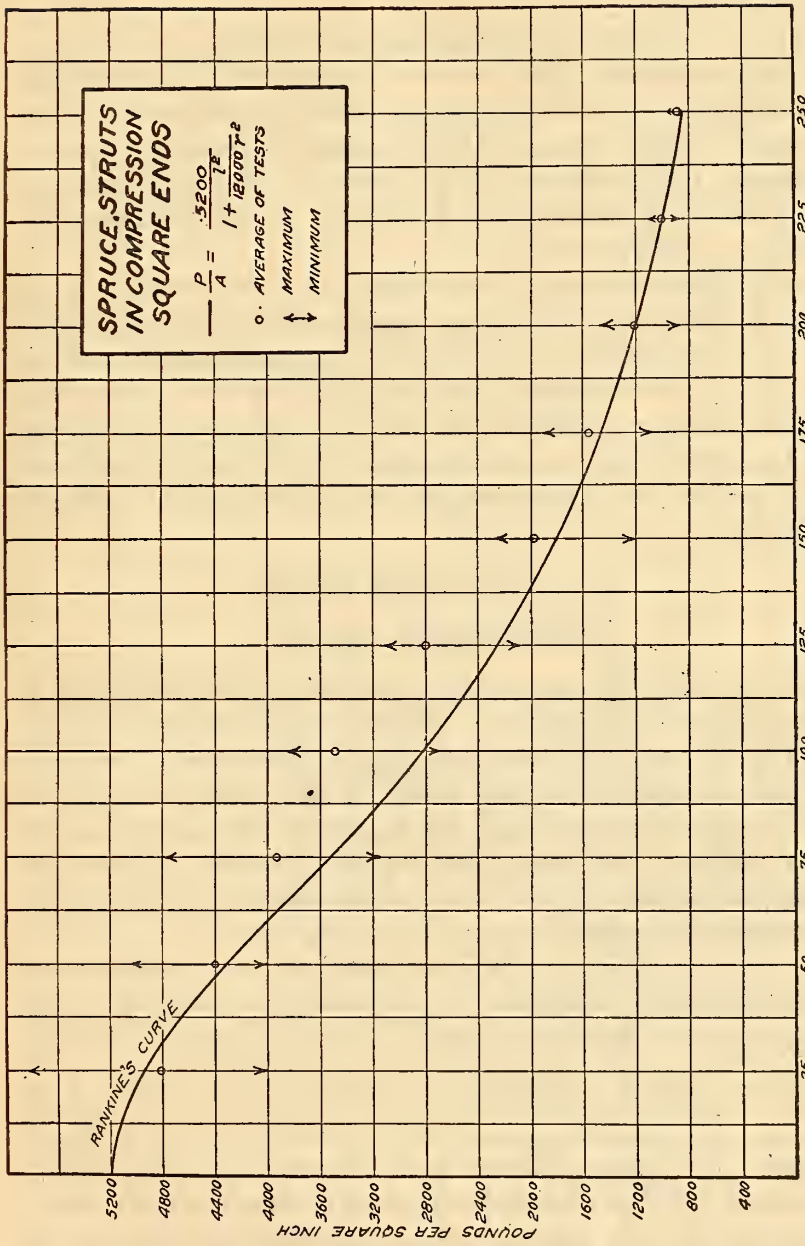

FIG. 5.—*Experimental results for square-end struts compared with Rankine's curve*

gives a fair approximation to the experiments except for the slenderness ratios from 75 to 150, within which range the error is on the side of safety.

It must be remembered that the heads of the Emery testing machine were practically fixed planes for these struts of small section and that the ends of the struts were carefully prepared. Where the ends of a strut rest on a flexible support, such as the spars of an airplane, this condition of approximately fixed ends does not obtain, and such struts should be treated as approximating the round-end condition.

A strut which is fastened to a rigid support by means of a pin frequently behaves as a round-end member in the plane of the pin. This was found to be true in the case of a stream-line strut which was tested. If it is desired that such a strut should behave as a fixed-end member, a stiffer connection and a longer and stiffer pin must be used. Also, since the moment at the ends of a fixed-end strut is as great as at the middle, the section should not be reduced at the ends.

IV. CROSS-BEND TESTS

1. EXPERIMENTAL RESULTS

One cross-bend test piece taken from each plank was loaded at the middle of a 24-inch span. The deflection was measured by means of an Ames dial under the point of application of the load. The dial was held by two rods which, in turn, rested on a pair of transverse pins placed through the beam over the supports at the neutral surface. For safety the dial was removed and the last readings taken with a steel scale. This accounts for the irregularity of the last points of the curves of Fig. 6.

The stress in the outer fibers has been plotted as ordinate and the corresponding deflection at the middle as abscissa in the curves of Fig. 6. The fiber stress, S, is calculated from the formula $S=\frac{Mc}{I}$, in which M is the bending moment at the middle of the span, I is the moment of inertia of the cross-section, and c is the distance of the extreme fibers from the neutral axis. The straight-line part of each curve has been extended to the deflection 0.2 inch in order to get the average slope from which to compute the modulus of elasticity (E). The deflection y at the middle of the span, is given by

$$y=\frac{PL^3}{48\ EI}$$

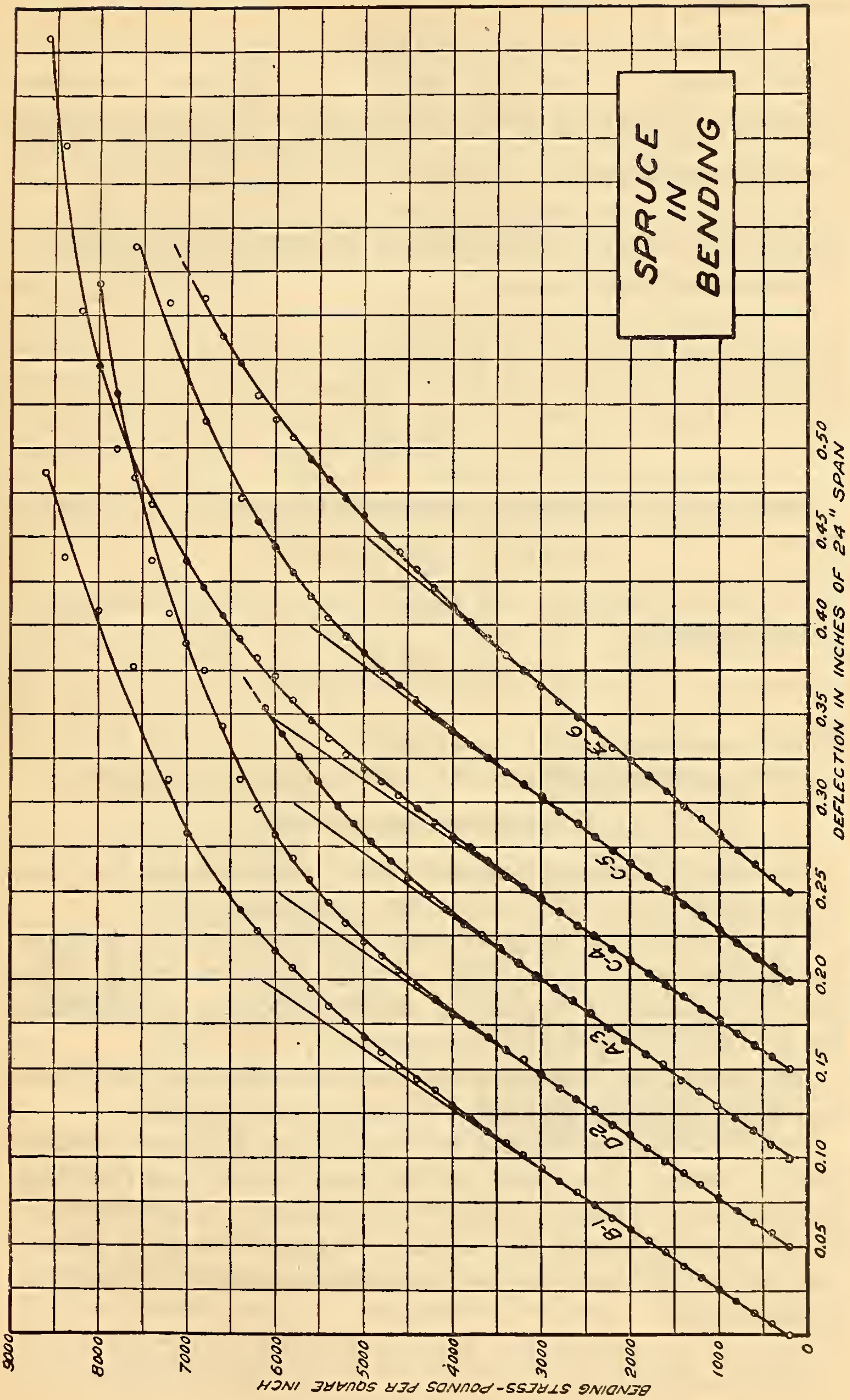

Fig. 6.—Deflection at middle and maximum computed unit stress for spruce beams loaded at middle

from which

$$E=\frac{PL^3}{48\ Iy}$$

in which P is the load at the middle and L is the length of the beam between supports. The moment at the middle is $\frac{PL}{4}$, and c for a rectangular section is $\frac{d}{2}$, if d is the depth of the section.

Substituting these values,

$$S=\frac{PL}{8}\frac{d}{I},$$

or

$$\frac{PL}{8\ I}=\frac{S}{d}.$$

Substituting this in the expression for the modulus of elasticity gives

$$E=\frac{SL^2}{6y\ d}.$$

This reduces to

$$E=\frac{480\ S}{d},$$

for $L = 24$ inches and $y = 0.2$ inch.

Table 4 gives a summary of the results of these calculations.

2. DISCUSSION OF RESULTS

The average modulus of elasticity from the six tests is 1 513 000 as compared with 1 566 000 from the compression tests.

The moduli of rupture given in this table are considerably below the average values found from tests of spruce specimens of this size. A modulus of rupture of from 10 000 to 12 000 pounds per square inch is frequently obtained.[1]

On the contrary, the moduli of elasticity computed from these data agree with average values obtained from other tests.

As the modulus of rupture does not appear in Euler's formula for the strength of columns, and the usual values were found for the modulus of elasticity, which does appear, it is believed that the formula developed here applies to spruce having a greater strength, as well as to the particular material used in this experimental work.

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[1] Bulletin 556—Mechanical Properties of Woods Grown in the United States, published by the Forest Products Laboratory, Madison, Wis.

Test piece E-6 was cross-grained and failed diagonally. Its modulus of elasticity was lower than that obtained from the other specimens from plank 6. On the other hand, B-1 gave a higher modulus of elasticity than most of the struts from plank 1. The difference between the values of the modulus of elasticity by the two methods is not large. Where it is not convenient to make compression tests of long struts, the modulus of elasticity may be determined from cross-bend tests; and this modulus may be used in Euler's formula.

If short blocks are tested in compression, the ultimate compressive strength thus obtained may be used with the modulus of elasticity from the bending tests to compute Ritter's constant. This constant, with the ultimate compressive strength, makes it possible to write a formula of the Rankine type.

V. ROUND-END STRUTS

1. EXPERIMENTAL RESULTS

The struts for the round-end tests were taken from three 16-foot planks numbered 7, 8, and 9. There were eight lengths from $\frac{L}{r} = 25$ to $\frac{L}{r} = 200$. The ends were finished in the same way as the struts for the square-end tests.

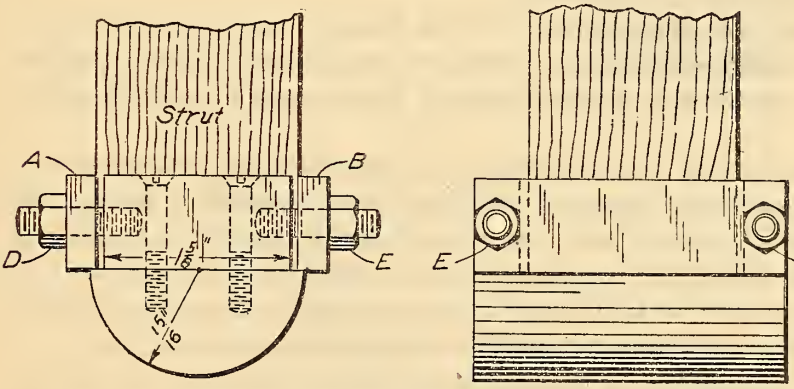

FIG. 7.—*Steel heads for round-end spruce struts*

To secure the round-end condition, the head shown in Fig. 7 was attached to each end. The ends of the struts were secured to steel half cylinders (diameter = 1 1/8 inches), which rested against the compression heads of the testing machine. They therefore rolled with little friction when the strut deflected.

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The cylinders were attached to the struts by means of the plates *A* and *B* held by the nuts *D*, *D'*, *E*, and *E'*. The strut was placed in the machine with the axes of the cylinders horizontal. If the strut showed considerable deflection in the vertical plane when a load of several hundred pounds was applied, the pressure was released to about 300 pounds and the position of the cylindrical bearing blocks shifted by turning the nuts. It was possible in this way to change the line of application of the loads as much as one-sixteenth of an inch on either side of the center. This insured axial loading and compensated for any lack of straightness in the strut.

Table 5 gives the results for one round-end strut. It will be noticed that the vertical deflection was small up to one-half the maximum load, and the horizontal deflection almost negligible until a comparatively large load was reached. At the total load of 3050 pounds the vertical deflection was 0.312 inch. Without compressing the strut further the deflection continued to increase and the load decreased. The load was again brought to 3050 pounds, and the deflection was found to be 0.482 inch. After the Howard gages and counterweights had been removed, the load was 2980 pounds. Further compression developed a maximum load of 3025 pounds, which under continued compression decreased to 3015 pounds.

This was characteristic of the behavior of round-end struts with small eccentricity. After the maximum load was reached there was a slow decrease of load, with a large increase of deflection.

The zero of the large Emery machine shifts with change of temperature, and, therefore, the beam was adjusted to zero at the beginning of each test. At the end of this test it was found that the zero had changed by an amount equivalent to a negative load of 30 pounds, so that the actual maximum was about 3020 pounds, and the maximum unit load was 990 pounds per square inch.

As this correction was known for the last values only, the stress-strain diagrams have been plotted from the loads as read, but the corrected values of the ultimate unit loads have been used in drawing the curves of ultimate strength and slenderness ratios. These corrections were not made in the case of the square-end struts.

2. TESTS OF FORMULAS

Fig. 8 shows the stress-strain diagrams and the horizontal and vertical deflections for the strut of Table 5. Diagrams were

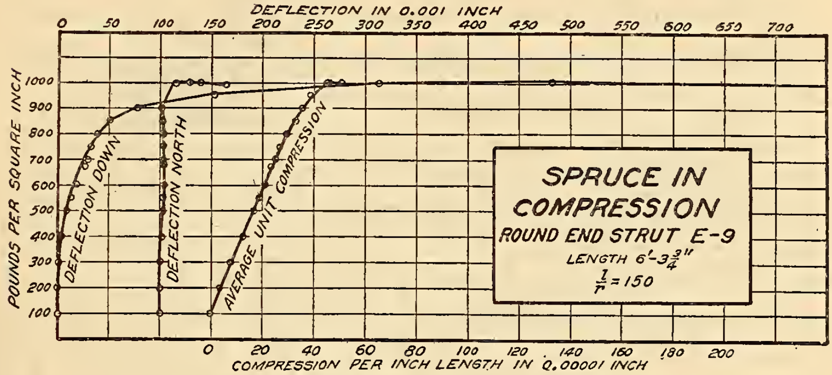

FIG. 8.—*Deflection and compression of round-end spruce strut*

plotted for each round-end strut similar to those of the square-end struts shown in Fig. 3. From these the modulus of elasticity and the proportional limit were determined for each strut. A summary is given in Table 6.

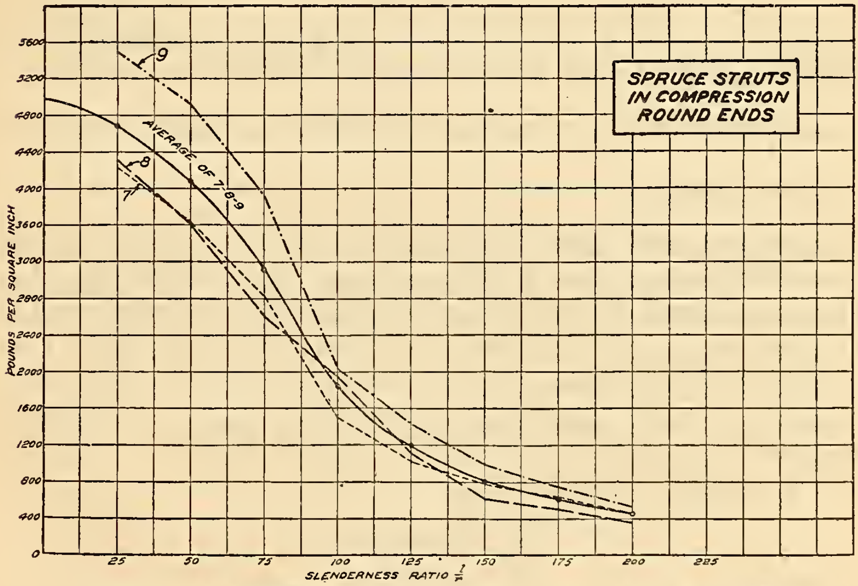

FIG. 9.—*Relation of ultimate strength of round-end spruce struts to slenderness ratio*

The curve of slenderness ratio and ultimate load for the struts from each plank, together with the average curve, is shown in Fig. 9. The average modulus of elasticity from all the tests is

1 910 000 pounds per square inch, which is large for spruce, and is due to the exceptionally large values of the modulus for all the struts from plank 9.

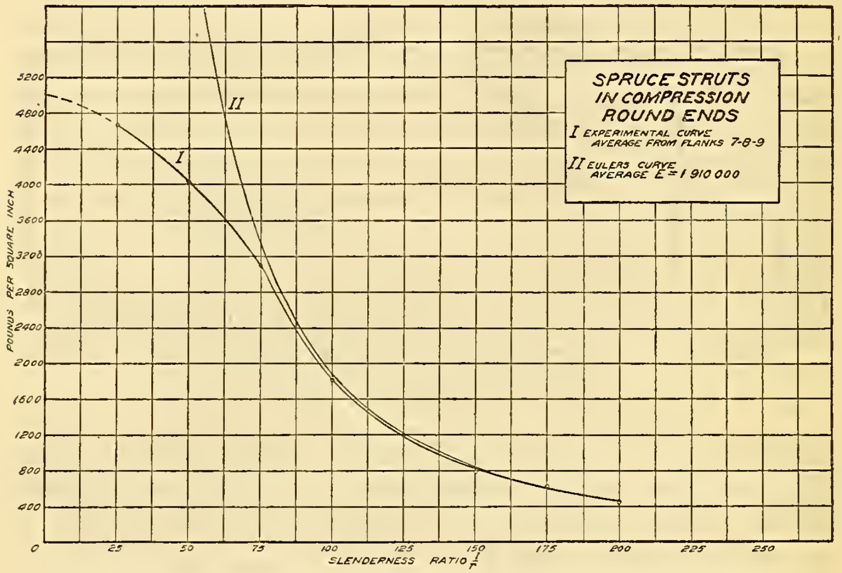

FIG. 10.—*Experimental curve for round-end spruce struts compared with Euler's*

In Fig. 10 the average curve is again shown, together with Euler's curve plotted for the average modulus of elasticity. These curves nearly coincide for values of $\frac{L}{r}$ greater than 100 and their difference is small for values of $\frac{L}{r}$ between 80 and 100. It is evident that Euler's formula may be used in the design of round-end timber struts if the slenderness ratio exceeds 100. In the case of spruce it is not advisable to use a modulus of elasticity as great as 1 900 000. Based on the results of these tests, the modulus should be taken as about 1 600 000. A value of 1 622 000 makes $\pi^2 E$ equal the round number 16 000 000. It is recommended that this value be used, and Euler's formula for round-end spruce struts then becomes

$$\frac{P}{A} = \frac{16\,000\,000}{\left(\frac{L}{r}\right)^2}.$$

This formula should be used for values of the slenderness ratio greater than 100, and may be used, with small error, for values between 80 and 100.

Fig. 11 gives the theoretical curve for round-end struts with slightly eccentric loads. It is plotted from the formula

$$S_u = \frac{P}{A}\left[1 + \frac{ec}{r^2} \cdot \sec\left(\sqrt{\frac{P}{AE}} \cdot \frac{L}{2r}\right)\right]$$

where S_u is the ultimate compressive strength of a short block, $\frac{P}{A}$ is the ultimate unit load on a given strut, c is the distance from the

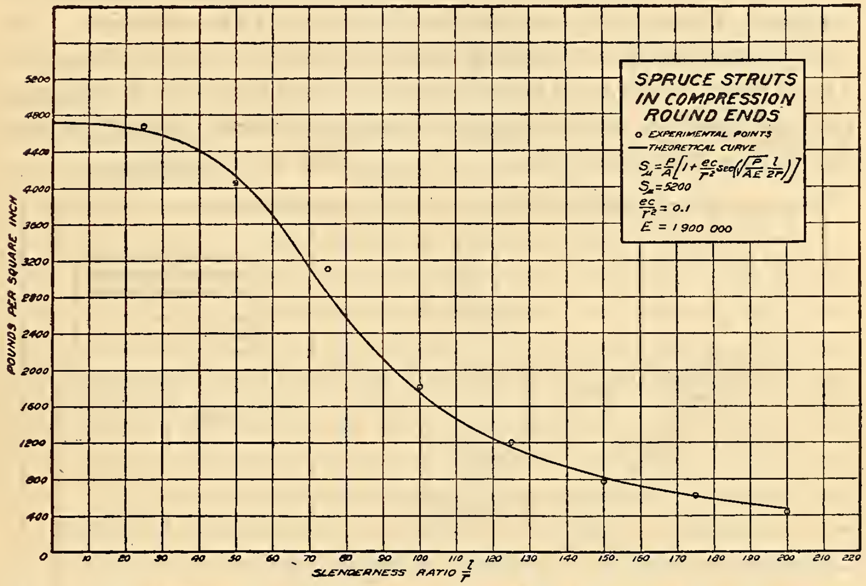

FIG. 11.—*Experimental results for round-end spruce struts compared with theoretical curve for a perfectly straight and uniform strut with round ends and slightly eccentric load*

center of gravity of the section to the extreme fiber, e is the eccentricity of the load, r is the radius of gyration, E is the modulus of elasticity, and L is the length of the strut. In calculating the data for this curve, S_u was taken as 5200 pounds per square inch, E as 1 910 000 pounds per square inch, and $\frac{ec}{r^2}$ assumed to be 0.1, which corresponds to an average eccentricity of $\frac{7}{240}$ inch for a section 1.75 inches square, free to turn about an axis parallel to one face, but not to turn about a diagonal of the section.

The small circles in Fig. 11 represent the the average experimental values of Table 6, from which curve *I* of Fig. 10 was drawn. From the close agreement of these points with the theoretical curve it is evident that it is only necessary to determine the ultimate strength of short blocks in compression and the modulus of

elasticity of the material in order to construct working curves for struts of all lengths. The modulus of elasticity may be determined from bending tests.

The amount of eccentricity is an uncertain element, and although a considerable difference in eccentricity makes little difference in the strength of long struts, it is always a questionable factor in the strength of short struts, no matter what formula is used.

The tests of long pieces by bending or compression should be carried to failure to determine the brashness of the material. The maximum loads in an airplane strut are impact loads (lasting for a brief time). For such loads, a wood of large modulus of resilience (as evidenced by a large deflection before rupture) should be preferred to a brasher material of even greater static strength.

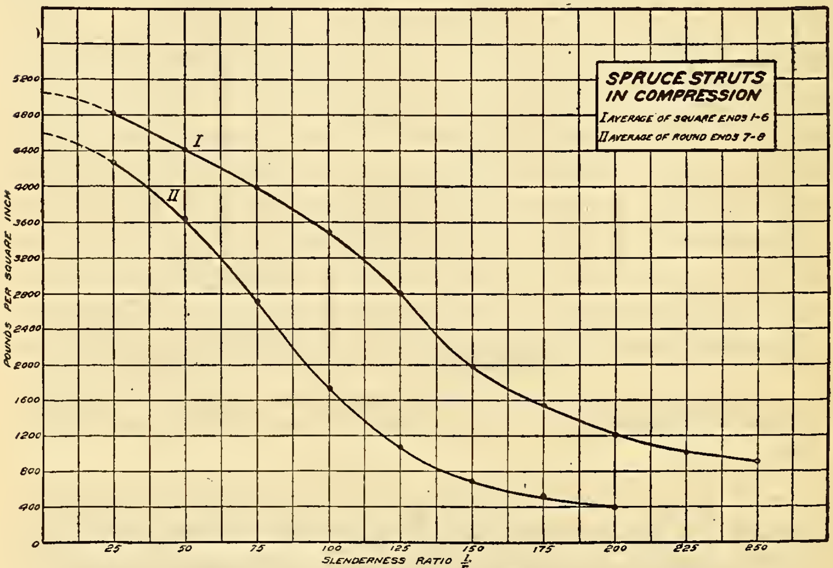

FIG. 12.—*Comparative ultimate strength of square-end and round-end struts*

Fig. 12 gives a comparison of round-end and square-end struts. Curve *I* is the mean of all the square-end struts, for which the average modulus of elasticity is 1 566 000 pounds per square inch. Curve *II* is the mean of the round-end struts from planks 7 and 8, for which the average modulus of elasticity is 1 670 000 pounds per square inch. The results from plank 9 are not included, as its large modulus increases the average of the round-end struts so as to vitiate the comparison.

VI. CONCLUSIONS

Euler's formula gives accurate results in the design of round-end spruce struts for values of $\frac{L}{r}$ greater than 100 and may be used with little error for values between 80 and 100.

For the ultimate strength per square inch for round-end spruce struts it is recommended that Euler's formula be written

$$\frac{P}{A}=\frac{16\,000\,000}{\left(\frac{L}{r}\right)^2}$$

which corresponds to a modulus of elasticity of 1 622 000 pounds per square inch.

For short round-end spruce struts use the Rankine formula

$$\frac{P}{A}=\frac{5200}{1+\frac{L^2}{3000r^2}}.$$

For square-end struts fitted accurately to rigid bodies, use the Rankine formula

$$\frac{P}{A}=\frac{5200}{1+\frac{L^2}{12000r^2}}.$$

If the ends are not well fixed, use values of the constant between

$$\frac{1}{12\,000} \text{ and } \frac{1}{3000}.$$

These formulas give the ultimate unit loads in pounds per square inch and must be divided by suitable factors of safety.

In the study of new materials a bending test may be made to determine the modulus of elasticity and the resilience, and a compression test of a short block to determine the ultimate compressive strength. The results thus obtained may be used in Euler's formula for round-end struts of uniform section having a slenderness ratio greater than 100 and in Rankine's formula for struts of all lengths. Instead of Rankine's formula, the theoretical equation of a strut with slightly eccentric load may be employed. This equation is more accurate than Rankine's formula, but the labor of computation is greater.

If long struts, having a slenderness ratio of 200 or more, be provided with round ends similar to Fig. 7, the modulus of elasticity may be determined quickly and accurately by means of

Euler's formula. Where a long compression machine is not available, a small strut may be tested by placing the lower end on a platform scale and applying pressure to the top by means of a lever until the buckling load is reached.

All timber for use in airplanes might be tested in this way as a part of the inspection. The pieces could be tested in the form of long strips before they are cut into short lengths. Where long strips are not available, a small test piece, not over 0.25 inch in thickness and of any convenient width, may be taken from each plank and tested to detruction as a round-end strut.

In all cases of square-end struts the amount of eccentricity is an uncertain factor, and the ends may seldom be regarded as fixed.

Ordinary airplane-pin connections should be regarded as round ends in the plane of the pins as well as in the plane perpendicular thereto.

VII. THEORY OF CERTAIN TAPERED STRUTS[2]

1. SHAPE OF STRUT

The results of the preceding experiments show that the load, under which a slender wood strut of uniform cross section fails, is determined primarily by the modulus of elasticity of the material. The following discussion develops the theory of the failure of a certain kind of tapered strut by elastic bending.

In one form of tapered strut the moment of inertia of any section between one end and the middle is proportional to the square of its distance from a fixed point beyond that end of the strut. Also, both halves of the strut are symmetrical with respect to a transverse plane at the middle. This form approximates many of the types used in practice. In Fig. 13, M is a point at a distance a to the left of the left end of the strut. The moment of inertia of any section in the left half, at a distance x from M, is given by

$$I = Cx^2,$$

where C is a constant depending upon the form of the strut. Likewise N is a point at a distance a to the right of the right end of the strut. The moment of inertia of any section in the right half is expressed by the same relation, with x measured to the left from N.

[2] A number of papers have been published in recent years on tapered struts. Arthur Morley (Engineering, **97**, pp. 566–568, and **104**, pp. 295–298), L. Bairstow and E. W. Stedman (Engineering, **98**, pp. 403–404), and John Case (Engineering **106**, pp. 295–298) have given methods of successive approximation, either graphic or analytic, for struts of arbitrarily varying cross section. W. M. Wallace (Engineering, **94**, pp. 831–832) has discussed by approximate methods columns whose moment of inertia varies according to an assumed analytical law. Since the present paper was in print, Akimasa Ono (Memoirs of the College of Engineering, Kyusha Imperial University, Fukuoka, Japan, **1**, No. 5, pp. 395–406) has given exact solutions for columns resembling those here discussed. In the latter two cases, however, the cross sections of the columns vanish at the end, whereas the columns here discussed have a finite cross section at the end.

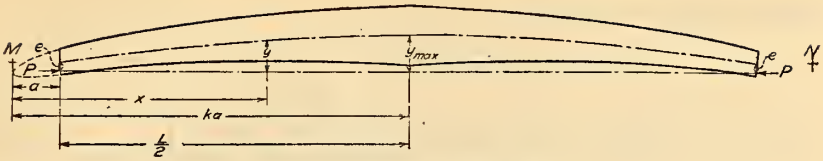

FIG. 13.—*Bent tapered strut*

2. DERIVATION OF FORMULAS

The load on this strut is P pounds acting at a distance e from the center of gravity of the end sections. Let y be the distance of the center of gravity of any section of the bent strut from the line of action of the load. The differential equation of the elastic curve for this strut is

$$EI\frac{d^2y}{dx^2} = -Py.$$

Substituting $I = Cx^2$, transposing and dividing by EC,

$$x^2\frac{d^2y}{dx^2} + \frac{Py}{EC} = 0.$$

This is a homogeneous linear differential equation of the second order. Its solution is written in three standard forms depending on the value of $\frac{4P}{EC}$.

$$\text{If } \frac{4P}{EC} < 1,\quad y = A_1\left(\frac{x}{a}\right)^{+\frac{1}{2}\sqrt{1-\frac{4P}{EC}}} + B_1\left(\frac{x}{a}\right)^{\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4P}{EC}}}$$

$$\text{If } \frac{4P}{EC} = 1,\quad y = \left(\frac{x}{a}\right)^{\frac{1}{2}}\left\{A_2 + B_2 \log\left(\frac{x}{a}\right)^{\frac{1}{2}}\right\}$$

$$\text{If } \frac{4P}{EC} > 1,\quad y = A\left(\frac{x}{a}\right)^{\frac{1}{2}} \sin\left\{B + \sqrt{\frac{4P}{EC} - 1}\, \log\left(\frac{x}{a}\right)^{\frac{1}{2}}\right\} \qquad (1)$$

These solutions may be verified by differentiating and substituting in the original equation. A_1, B_1, A_2, B_2, and A are integration constants having the dimensions of a length, and B is a dimensionless integration constant.

The constant C is generally a small quantity and for loads which cause failure the fraction $\frac{4P}{EC}$ is greater than unity, so that only the third form of the solution need be used. To determine the constants in equation (1) we have the boundary conditions: At the end of the strut $x = a$, $y = e$, at the middle of the strut $x = ka$, $y = y_{\text{max}}$, $\frac{dy}{dx} = o$, $k = 1 + \frac{L}{2a}$.

Substituting in equation (1)

$$e=A\ \sin\ B$$

$$y_{\max}=Ak^{\frac{1}{2}}\sin\left(B+\sqrt{\frac{4P}{EC}-1}\ \log\ k^{\frac{1}{2}}\right)$$

$$=Ak^{\frac{1}{2}}\sin\ \alpha \tag{2}$$

Differentiating equation (1)

$$\frac{dy}{dx}=\frac{1}{2a}A\left(\frac{x}{a}\right)^{-\frac{1}{2}}\ \sin\left(B+\sqrt{\frac{4P}{EC}-1}\log\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)$$
$$+\sqrt{\frac{4P}{EC}-1}\cos\left(B+\sqrt{\frac{4P}{EC}-1}\log\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)$$

Substituting $x=ka$ and equating to zero

$$\frac{1}{2a}Ak^{-\frac{1}{2}}\left[\sin\alpha+\sqrt{\frac{4P}{EC}-1}\cos\ \alpha\right]=0$$

or

$$\tan\ \alpha=-\sqrt{\frac{4P}{EC}-1}$$

Then

$$B=\alpha-\sqrt{\frac{4P}{EC}-1}\ \log k^{\frac{1}{2}}$$

and

$$A=\frac{e}{\sin B}$$

Substituting in equation (2)

$$y_{\max}=\frac{ek^{\frac{1}{2}}\sin\alpha}{\sin B}$$

$$=\frac{ek^{\frac{1}{2}}\sin\alpha}{\sin\alpha\cos\left(\sqrt{\frac{4P}{EC}-1}\ \log\ k^{\frac{1}{2}}\right)-\cos\alpha\sin\left(\sqrt{\frac{4P}{EC}-1}\ \log\ k^{\frac{1}{2}}\right)}$$

or

$$y_{\max}=\frac{ek^{\frac{1}{2}}}{\cos\left(\sqrt{\frac{4P}{EC}-1}\ \log\ k^{\frac{1}{2}}\right)+\frac{1}{\sqrt{\frac{4P}{EC}-1}}\sin\left(\sqrt{\frac{4P}{EC}-1}\ \log k^{\frac{1}{2}}\right)} \tag{3}$$

3. APPLICATION OF FORMULAS

This equation may be used to calculate the maximum fiber stress in the strut under any known eccentric load.

Use the formula,

$$S=\frac{P}{A}+\frac{Mc}{I}$$

where A is the area of the section, M is the bending moment at the section, and c is the distance of the extreme fibers from the center of gravity of the section. In the equations of a strut $M=Py$ if y is measured from the line of the load, as in Fig. 13. At the ends of the strut $M=Pe$, at the middle $M=Py_{max}$, at the other sections $M=Py$ where y_{max} and y may be calculated from equations (3) and (2), respectively. Generally it is only necessary to calculate the stress at the middle and at the ends of the strut. In a strut for which $I=Cx^2$ the maximum stress occurs a short distance from the middle. The difference between the maximum stress and the stress at the middle section is not great.

The maximum safe load—that is, the load which gives the maximum safe fiber stress—can thus be determined by a process of trial and error.

For relatively slender struts a simpler method may be used. If the load is assumed to be centrally applied ($e=o$) and P so chosen as to make the denominator in equation (3),

$$\cos\left(\sqrt{\frac{4P}{EC}-1}\ \log k^{\frac{1}{2}}\right)+\frac{1}{\sqrt{\frac{4P}{EC}-1}}\sin\left(\sqrt{\frac{4P}{EC}-1}\ \log k^{\frac{1}{2}}\right) \qquad (4)$$

equal to zero, or in another form to make

$$\tan\alpha\ \log k^{\frac{1}{2}}=-\alpha \qquad (5)$$

the deflection y become indeterminate as in a strut of uniform cross section under Euler's load. Equation (5) is, in fact, a generalization of Euler's formula for this type of tapered strut and, as will be shown later, reduces to Euler's formula when $k=1$.

In practice the eccentricity is seldom known, but in slender struts to which these equations are applicable the effect of eccentricity is small and the ultimate load in practice is only slightly less than that given by equation (5).

From inspection of the equation it is evident that the critical load is that which makes the angle $\left(\sqrt{\frac{4P}{EC}-1}\ \log k^{\frac{1}{2}}\right)=-\tan\alpha$ $\log k^{\frac{1}{2}}$ a little greater than $\frac{\pi}{2}$. The cosine will then be negative and the sine positive.

To determine whether a given load is safe, multiply the load by a suitable factor of safety and substitute in expression (4). If the angle $\left(\sqrt{\frac{4P}{EC}-1}\log k^{\frac{1}{2}}\right)$ is less than $\frac{\pi}{2}$ radians, the load is a safe one. If the angle is greater than $\frac{\pi}{2}$ radians, the load is still safe provided expression (4) is positive. If it is negative, the load is too large.

Before applying these equations it is necessary to express C and k in terms of the dimensions of the strut. Let I_1 be the moment of inertia at the end (where $x=a$), and let I_2 be the moment of inertia at the middle (where $x=ka$).

$$I_1=C\,a^2;\qquad I_2=C(ka)^2;$$

$$k^2=\frac{I_2}{I_1}.$$

Since

$$ka=a+\frac{L}{2},$$

$$\sqrt{I_1}=\sqrt{C}\,a;\qquad \sqrt{I_2}=\sqrt{C}\left(a+\frac{L}{2}\right);$$

$$\sqrt{C}=\frac{\sqrt{I_2}-\sqrt{I_1}}{\frac{L}{2}}.$$

If all sections of the strut are similar figures, so that I varies as the fourth power of the transverse dimensions,

$$k=\frac{D_2^2}{D_1^2};\quad k^{\frac{1}{2}}=\frac{D_2}{D_1},$$

where D_2 and D_1 are homologous transverse dimensions at the middle and ends of the strut, respectively.

The maximum load which a given strut will carry may be found by the method of trial and error. This method of applying the formula is shown in the example which follows. The expression (4) is used instead of equation (5) because the interpolation is more nearly linear.

EXAMPLE.—A wooden strut is 6 feet long, 2 inches square at the middle, 1.6 inches square at the ends, and tapers according to the law $I=Cx^2$. If the modulus of elasticity $(E)=1\,600\,000$ pounds per square inch, find the ultimate load.

$$k^{\frac{1}{2}}=\frac{D_2}{D_1}=\frac{2}{1.6}=\frac{5}{4}$$

$$\log_e k^{\frac{1}{2}} = 1.60944 - 1.38629 = 0.22315$$

$$I_1 = \frac{1.6^4}{12}; \quad I_2 = \frac{2^4}{12}; \quad \sqrt{C} = \frac{2^2 - 1.6^2}{\sqrt{12} \times 36}; \quad C = \frac{(4-2.56)^2}{12 \times 36^2} = \frac{1}{7500}$$

By Euler's formula the ultimate strength of a uniform strut 2 inches square and 72 inches long is

$$P = \frac{\pi^2 \times 1\ 600\ 000 \times 16}{72 \times 72 \times 12} = 4061.5 \text{ pounds.}$$

The load on the tapered strut must be considerably smaller. First try 3000 pounds.

$$\frac{4P}{EC} = \frac{3000 \times 4 \times 7500}{1\ 600\ 000} = 56.25$$

$$\sqrt{\frac{4P}{EC} - 1} = \sqrt{55.25} = 7.433$$

$$7.433 \times 0.22315 = 1.6605 \text{ radians} = 95^\circ\ 8'$$

$$\text{Cos } 95^\circ\ 8' + \frac{\sin 95^\circ\ 8'}{7.433} = -0.0895 + 0.1338 = 0.0443$$

The load of 3000 pounds is below the ultimate.
Next try 3200 pounds.

$$\sqrt{\frac{4P}{EC} - 1} = \sqrt{59} = 7.6811$$

$$\frac{7.6811 \times 0.22315 \times 180}{\pi} = 98.205^\circ = 98^\circ\ 12'$$

$$\text{Cos } 98^\circ\ 12' + \frac{\sin 98^\circ\ 12'}{7.6811} = -0.1426 + 0.1289 = -0.0137$$

This load is too great. Interpolating

$$P = 3000 + \frac{443 \times 200}{443 + 137} = 3153$$

Recalculating with $P = 3153$ pounds,

$$\sqrt{\frac{4P}{EC} - 1} = 7.6236; \ \theta = 97^\circ\ 28'$$

$$\text{Cos } 97^\circ\ 28' + \frac{\sin 97^\circ\ 28'}{7.6236} = -0.1299 + 0.1301 = 0.0002$$

Interpolating again,

$$P = 3153 + \frac{2 \times 47}{139} = 3153.7 \text{ pounds}$$

The ratio of this load to Euler's ultimate load on a uniform strut 2 inches square is $\frac{3153.7}{4061.5} = 0.7765$. We will call this the strength ratio of the strut. It will be shown later that this ratio holds for a strut of any length and material for which $k^{\frac{1}{2}} = 1.25$, provided the slenderness ratio is such that Euler's formula is applicable for the strut of uniform section.

While it is not practicable to solve equation (5) for P when k is given, except by the method of trial and error, it is easy to find the value of k corresponding to any given figure for $\frac{4P}{EC}$ and from a series of such values plot a curve which will simplify the computations.

Solving equation (5) for (κ), $\log_e k = \frac{2\alpha}{\tan \alpha}$ (6)

where $\tan \alpha = \sqrt{\frac{4P}{EC} - 1}$ which determines $\frac{4P}{EC}$ as a function of k. The first part of Table 7 gives the calculation for k by logarithms.

Strength Ratio.—The ratio between the ultimate load of this type of tapered strut and Euler's ultimate load of a strut of uniform cross section whose length is the length of the tapered strut and whose moment of inertia is the maximum moment of inertia of the tapered strut, is independent of the length and moment of inertia and depends for its value on k alone. This ratio will be called the strength ratio of the tapered strut. To calculate this ratio, it is necessary to express C in terms of I_2, L, and k and substitute in $\frac{4P}{EC}$, which is known from equation (6) or Table 7 as a function of k.

$$\sqrt{C} = \frac{\sqrt{I_2} - \sqrt{I_1}}{\frac{L}{2}} = \frac{\sqrt{I_2} - \frac{\sqrt{I_2}}{k}}{\frac{L}{2}} = 2\frac{\sqrt{I_2}}{L}\left(1 - \frac{1}{k}\right)$$

$$C = \frac{4I_2}{L^2}\left(1 - \frac{1}{k}\right)^2$$

Substituting this value of C,

$$\frac{4P}{EC} = \frac{PL^2}{EI_2\left(1 - \frac{1}{k}\right)^2}$$

from which,

$$\frac{PL^2}{EI_2} = \frac{4P}{EC}\left(1 - \frac{1}{k}\right)^2 \qquad (7)$$

By Euler's formula the ultimate load of a strut of uniform section of length L and moment of inertia I_2 is,

$$P' = \frac{\pi^2 E I_2}{L^2},$$

from which

$$\frac{P' L^2}{E I_2} = \pi^2. \tag{8}$$

Dividing (7) by (8), gives the strength ratio of the tapered strut

$$\frac{P}{P'} = \left(\frac{4P}{EC}\right) \cdot \frac{\left(1 - \frac{1}{k}\right)^2}{\pi^2}$$

which is seen to be a function of k alone, as stated above.

The second part of Table 7 gives most of the steps of the calculations of the strength ratios for the values of $\frac{4P}{EC}$ in the first part of the table. It is to be noted that as k approaches unity the strength ratio approaches unity showing that equation (5) reduces to Euler's formula for $k = 1$. The last column of the table gives $\sqrt{\frac{1}{k}}$.

$$\sqrt{\frac{1}{k}} = \left(\frac{I_1}{I_2}\right)^{\frac{1}{4}}$$

If all sections of the strut are similar figures,

$$\sqrt{\frac{1}{k}} = \frac{D_1}{D_2},$$

D_1 and D_2 being homologous transverse dimensions at the ends and middle of the strut, respectively.

The curve of Fig. 14 is plotted from Table 7, and gives the strength ratio for values of $\sqrt{\frac{1}{k}}$ above 0.2. Lower values are omitted as such struts will generally fail by compression at the ends. This curve affords an easy method of calculating the ultimate strength of a slender tapered strut for which the equation is $I = Cx^2$. Calculate the strength of a uniform strut of the same length and of cross section equal to the maximum section of the tapered strut and multiply this result by the strength ratio as read from the curve.

4. APPROXIMATE CALCULATIONS

The straight line of Fig. 14, at an angle of 45°, does not deviate greatly from the strength-ratio curve. For approximate calculations the strength ratio may be taken as equal to $\sqrt{\frac{1}{k}}$.

If all sections are similar figures, $\sqrt{\frac{1}{k}}=\frac{D_1}{D_2}$, and the strength of the tapered strut is to the strength of the uniform strut as a transverse dimension at the end is to the homologous transverse dimension at the middle.

Another approximate solution is by means of the moment of inertia at some point between the end and the middle. We will consider the section at one-third the length from the end; for the case, $\sqrt{\frac{1}{k}}=0.5$.

$$\frac{1}{k}=0.25;\quad k=4;\quad ka-a=3a=\frac{L}{2}.$$

$$x=a+\left(\frac{2}{3}3a\right)=3a.$$

$$I=Cx^2=9Ca^2.$$

$$I_2=C(ka)^2=16Ca^2.$$

$$\frac{I}{I_2}=\frac{9}{16}=0.5625.$$

The moment of inertia at one-third the length from the end is 56 per cent of that at the middle. If the moment of inertia at this section be taken as the average I for use in Euler's formula, it will give an apparent strength ratio of 0.56. Fig. 14 shows that the real strength ratio is 0.49, an error of about 14 per cent in this case.

The second column of Table 8 gives the ratio of the moment of inertia at the third points to that at the middle for a series of values of $\sqrt{\frac{1}{k}}$. A comparison of these values, with the strength ratios in the last column, shows that the moment-of-inertia method may be used with little error for struts of moderate taper. Greater accuracy may be secured by taking, for any given value of moment-of-inertia ratio, the corresponding strength ratio from the last column of Table 8.

The third column of Table 8 gives the ratio of the moment of inertia at five-sixteenths of the length from the end, to the moment of inertia at the middle. These agree more closely with the strength ratios and err slightly on the side of safety in the case of struts of small taper.

These figures apply strictly to struts for which $I=Cx^2$. Actual struts taper less rapidly near the middle, so that the error in using the moment of inertia at the one-third point may be a little greater than indicated by Table 8.

5. RELATION OF THE STRENGTH OF A STRUT TO ITS STIFFNESS AS A BEAM

The approximate methods already given apply strictly to struts for which $I=Cx^2$, and with slight error, to other forms. In general, the ultimate strength of a relatively slender strut varies with its stiffness. In order to show how nearly the ultimate strength of a strut is proportional to its stiffness as a beam, we will calculate the deflection of some tapered struts, supported at the ends and loaded at the middle.

Using the method of internal work,

$$U=\int\frac{M^2}{2EI}\,dx,$$

in which U is the work and M is the bending moment. For a beam supported at the ends and loaded at the middle, with the origin at a distance a to the left of the left end,

$$M=\frac{P}{2}\,(x-a).$$

$$U=\int\frac{P^2}{8EI}\,(x-a)^2\,dx=\frac{P^2}{8EC}\int\left(1-\frac{2a}{x}+\frac{a^2}{x^2}\right)dx$$

$$U=\frac{P^2}{8EC}\left[x-2a\log\left(\frac{x}{a}\right)-\frac{a^2}{x}\right]_a^{ka},$$

for one-half the beam. For the entire beam

$$U=\frac{P^2a}{4EC}\,\left(k-\frac{1}{k}-2\log k\right).$$

The external work is $\frac{P}{2}\,y_{max}$. Equating the two expressions for work,

$$y_{max}=\frac{Pa}{2EC}\,\left(k-\frac{1}{k}-2\log k\right).$$

Substituting $\frac{P}{EC}=\frac{PL^2}{4EI_2\left(1-\frac{1}{k}\right)^2}$ and $a=\frac{L}{2\,(k-1)}$,

$$y_{\max}=\frac{P\,L^3\,k^2}{16EI_2\,(k-1)^3}\left(k-\frac{1}{k}-2\log k\right).$$

The deflection at the middle of a beam of uniform section, having moment of inertia I_2 is

$$y_{\max}=\frac{PL^3}{48EI_2},$$

if the beam is supported at the ends and loaded at the middle. Since stiffness is inversely proportional to the deflection due to equal loads, the relative stiffness is given by

$$\text{relative stiffness}=\frac{(k-1)^3}{3k^2\left(k-\frac{1}{k}-2\log k\right)}.$$

For the case in which $k=4$ and $\sqrt{\frac{1}{k}}=0.5$,

$$\text{relative stiffness}=\frac{27}{48\,(4-0.25-2\times 1.38629)}=0.577.$$

The stiffness of a beam with this taper is 58 per cent of that of a uniform beam of the same length and maximum moment of inertia. From Table 8 it is seen that the strength ratio for this value of k is 0.49.

The fourth column of Table 8 gives a series of stiffness ratios. These are somewhat higher than the corresponding strength ratios, and the difference is greater in the struts of large taper.

These computations are for struts for which $I=Cx^2$, which approximates the form of actual struts, except that the surfaces of the two halves meet at a slight angle at the middle, instead of on a common tangent. A formula was derived for struts having the equation $I=I_2-Cx^2$, in which x is the distance from the middle. This strut has the desired form at the middle, but converges more slowly near the middle and more rapidly near the ends than the actual struts. The calculations show higher strength ratios and stiffness ratios than those for the struts of Table 8. The stiffness ratios exceed the strength ratios in about the same proportion in both cases.

To find approximately the strength of any tapered strut, support it at the ends as a beam, apply a light load at the middle, and measure the deflection y_{max}. From the equation of a uniform beam,

$$y_{max} = \frac{PL^3}{48EI},$$

calculate the equivalent EI. Substitute this value of EI in Euler's formula to get the ultimate strength. The result will be somewhat too great, since the stiffness ratio exceeds the strength ratio.

For greater accuracy test a uniform strut of the same material of cross section equal to that at the middle of the tapered strut. From the two deflections get the stiffness ratio and make the corresponding relative correction in the value of EI. The loads and deflections must be small so as not to injure the strut.

Example.—A tapered strut 5 feet long supported at the ends is deflected 0.060 inch at the middle by a load of 10 pounds at the middle.

$$EI = \frac{10 \times 60^3}{48 \times 0.060} = 750\,000$$

Suppose a uniform beam of the same length is deflected 0.042 inch by the same load. The stiffness ratio is $\frac{42}{60} = 0.70$. Interpolating from Table 8, the corresponding strength ratio is found to be 0.62. The corrected value of the equivalent EI is

$$750\,000 \times \frac{62}{70} = 660\,000.$$

Instead of measuring the deflection of a uniform strut, the value of $\left(\frac{I_1}{I_2}\right)^{\frac{1}{4}}$ may be calculated from the dimensions of the strut and used in Table 8 to find the relation of strength ratio to stiffness ratio.

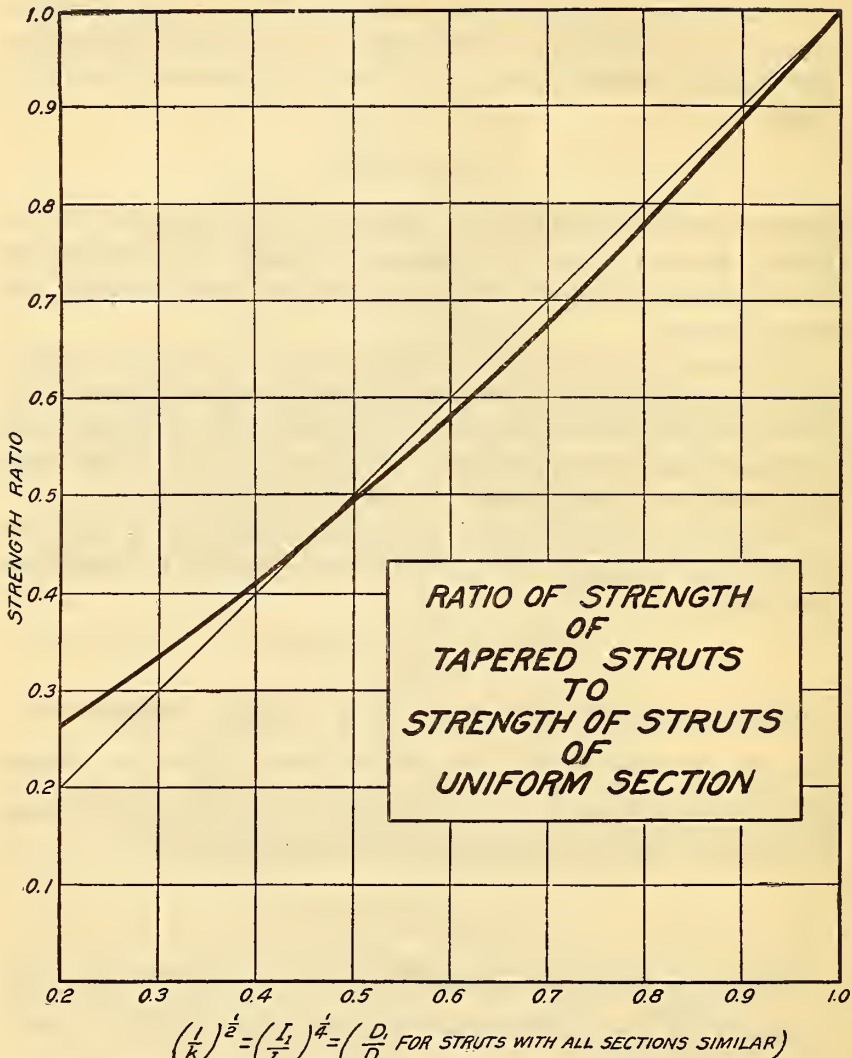

FIG. 14.—*Comparison of strength of tapered struts and struts of uniform section*

TABLE 1.—Compression Test Square-End Strut B–4, Tested Aug. 20, 1917, by J. E. B.

[Length, 4 feet 2½ inches; breadth, 1.738 inches; depth, 1.762 inches. Slenderness ratio, 100. Compression with 30-inch Howard gages placed 2.75 inches apart. Counterweight of 9 pounds at each end of gages. Deflection measurements taken with Johnson dials.]

Unit stress, pounds per square inch	Compression, inches						Average unit compression, inch per inch	Deflection, inches	
	North gage			South gage					
	At gage	At surface		At gage	At surface			Up	South
		In 30 inches	Per inch		In 30 inches	Per inch			
I	II	III	IV	V	VI	VII	VIII	IX	X
200	0.0000	0.00000	0.000000	0.0000	0.00000	0.000000	0.000000	0.000	0.000
400	.0049	.00465	.000155	.0036	.00384	.000128	.000142	.004	.003
600	.0090	.00873	.000291	.0075	.00777	.000259	.000275	.009	.003
800	.0122	.01189	.000396	.0105	.01081	.000360	.000378	.009	.004
1000	.0163	.01595	.000632	.0144	.01475	.000492	.000512	.010	.005
1200	.0201	.01961	.000654	.0174	.01789	.000596	.000625	.012	.007
1400	.0245	.02390	.000797	.0212	.02180	.000727	.000762	.013	.009
1600	.0284	.02767	.000922	.0244	.02513	.000838	.000880	.013	.012
1800	.0330	.03215	.001072	.0283	.02915	.000972	.001022	.013	.015
2000	.0372	.03618	.001206	.0316	.03262	.001087	.001147	.013	.018
2200	.0414	.04015	.001338	.0345	.03475	.001192	.001265	.020	.021
2400	.0461	.04466	.001489	.0382	.03964	.001321	.001405	.021	.021
2600	.0510	.04927	.001642	.0415	.04323	.001441	.001541	.021	.030
2800	.0563	.05414	.001804	.0444	.04656	.001552	.001678	.022	.036
3000	.0611	.05897	.001952	.0472	.04973	.001658	.001805	.022	.043
3100	.0639	.06105	.002035	.0482	.05105	.001702	.001868	.024	.048
3200	.0683	.06495	.002165	.0499	.05324	.001775	.001970	.025	.056
3400	.0742	.07002	.002334	.0512	.05538	.001846	.002090	.029	.071
3500[a]	.0810	.07560	.002520	.0513	.05670	.001890	.002205	.033	.089
3446								.045	.113
3560								.048	.122
3634								.055	.146
3725[b]									
3520[c]								.120	.275
3447[d]								.455	.271

[a] Removed Howard gages and counterweights. Load dropped to 3446 pounds per square inch.
[b] Maximum load. Continued to deflect south with compression head stationary.
[c] Cracking noise. Strut bent up and south.
[d] Compression failure, 1.5 inches from middle.

TABLE 2.—Compression Test Square-End Strut C–4, Tested Aug. 17, 1917, by J. E. B.

[Length, 5 feet 3⅛ inches; breadth, 1.738 inches; depth, 1.763 inches. Slenderness ratio, 125. Tested on large Emery machine. Compression measured with 30-inch Howard gages placed 2.75 inches apart. Counterweights of 9 pounds at each end of gages. Deflection measurements taken with Johnson dials.]

Unit stress, pounds per square inch	Compression, inches						Average unit compression, inch per inch	Deflection, inches	
	North gage			South gage					
	At gage	At surface		At gage	At surface			Up	South
		In 30 inches	Per inch		In 30 inches	Per inch			
I	II	III	IV	V	VI	VII	VIII	IX	X
200	0.0000	0.00000	0.00000	0.0000	0.00000	0.00000	0.00000	0.000	0.000
400	.0038	.00380	.00127	.0038	.00380	.00127	.00127	.002	.000
600	.0075	.00740	.00247	.0070	.00710	.00237	.00242	.005	.000
800	.0114	.01133	.00378	.0110	.01107	.00369	.00373	.006	.000
1000	.0146	.01449	.00483	.0140	.01411	.00470	.00477	.006	.000
1200	.0186	.01845	.00615	.0178	.01795	.00598	.00607	.011	.000
1400	.0222	.02211	.00737	.0217	.02179	.00726	.00732	.011	.000
1600	.0252	.02502	.00834	.0242	.02438	.00813	.00823	.011	.000
1800	.0296	.02935	.00978	.0282	.02845	.00948	.00963	.012	.003
2000	.0333	.03305	.01102	.0319	.03215	.01072	.01087	.013	.005
2200	.0379	.03754	.01251	.0359	.03626	.01209	.01230	.013	.005
2400	.0412	.04084	.01361	.0392	.03954	.01316	.01340	.015	.005
2600	.0456	.04513	.01504	.0430	.04347	.01449	.01477	.016	.004
2700	.0471	.04668	.01556	.0448	.04522	.01507	.01532	.019	.004
2800	.0494	.04896	.01632	.0470	.04744	.01561	.01607	.020	.004
2900	.0510	.05067	.01689	.0492	.04953	.01651	.01670	.022	.004
3000	.0524	.05240	.01747	.0522	.05220	.01740	.01743	.024	.004
3100 [a]	.0520	.05320	.01773	.0586	.05740	.01913	.01843	.022	.028

[a] Removed gages and counterweights. Load dropped to 2840 pounds per square inch. Strut suddenly bent upward and continued to bend with reduced load. Compression failure, 1 inch from middle at bottom and one side.

TABLE 3.—Summary of Tests of Square-End Struts

Strut	Length	Breadth	$\frac{L}{r}$	Depth	$\frac{L}{r}$	Area	Modulus of elasticity	Proportional limit	Ultimate unit load
	Inches	Inches		Inches		Inches²	Lbs./in.²	Lbs./in.²	Lbs./in.²
A-1	12.625	1.745	25.1	1.765	24.8	3.08	1 840 000	2800	5000
E-2	12.625	1.736	25.2	1.745	25.1	3.03	1 670 000	1400	4191
C-3	12.625	1.745	25.1	1.765	24.8	3.08	1 870 000	2000	4026
D-4	12.625	1.742	25.1	1.762	24.8	3.07	1 530 000	2200	4400
E-5	12.625	1.742	25.1	1.775	24.6	3.09	1 400 000	2800	5227
A-6	12.625	1.746	25.0	1.735	25.2	3.03	1 830 000	4000	5809
C-5	12.625	1.757	24.9	1.733	25.3	3.04	1 690 000	1800	4375
E-6	12.625	1.750	25.0	1.753	25.0	3.07	1 730 000	2800	5472
Average									4812
C-2	25.250	1.726	50.7	1.731	50.5	2.99	1 650 000	2200	4047
D-3	25.250	1.745	50.1	1.756	49.8	3.06	1 700 000	2000	4248
E-4	25.250	1.744	50.2	1.764	49.6	3.08	1 760 000	2800	4383
A-5	25.250	1.737	50.4	1.765	49.6	3.07	1 600 000	3200	4723
B-6	25.250	1.753	49.9	1.755	49.8	3.08	1 350 000	2800	5032
Average									4487
C-1	37.875	1.750	75.0	1.732	75.8	3.03	1 140 000	1600	3168
E-3	37.875	1.745	75.2	1.766	74.3	3.08	1 500 000	1800	3743
A-4	37.875	1.740	75.4	1.758	74.7	3.06	1 390 000	2200	3400
B-5	37.875	1.737	75.8	1.772	74.0	3.08	1 860 000	2600	4580
C-6	37.875	1.747	75.1	1.748	75.1	3.05	1 400 000	2400	3734
B-2a	37.875	1.723	76.1	1.722	76.2	2.97	1 770 000	2200	4417
B-2b	37.875	1.761	74.6	1.732	75.8	3.05	1 480 000	2600	4764
C-3	37.875	1.755	74.8	1.738	75.5	3.05	1 680 000	2200	4033
Average									3979
D-1	50.500	1.732	101.0	1.746	100.2	3.02	1 180 000	1800	2748
E-2	50.500	1.735	100.6	1.729	101.2	3.00	1 600 000	2100	3300
A-3	50.500	1.768	98.7	1.744	100.3	3.08	1 600 000	2400	3864
B-4	50.500	1.738	100.7	1.762	99.3	3.06	1 560 000	2600	3725
D-6	50.500	1.745	100.2	1.765	99.1	3.08	1 540 000	2600	3815
Average									3490
E-1	63.125	1.734	126.0	1.754	124.7	3.04	1 140 000	1600	2100
A-2	63.125	1.736	126.0	1.752	124.8	3.04	1 770 000	2000	2960
B-3	63.125	1.770	123.6	1.743	125.5	3.09	1 650 000	2200	2913
C-4	63.125	1.738	125.8	1.763	124.0	3.06	1 640 000	2600	3100
D-5	63.125	1.738	125.8	1.763	124.0	3.06	1 820 000	1800	2931
Average									2801
E-1	75.750	1.735	151.2	1.746	150.3	3.03	800 000	1000	1200
A-2	75.750	1.738	151.0	1.758	149.2	3.06	1 720 000	1500	1993
B-3	75.750	1.762	148.9	1.746	150.3	3.08	1 720 000	2000	2250
D-5	75.750	1.738	151.0	1.748	150.1	3.03	1 520 000	1750	2178
E-6	75.750	1.750	150.0	1.754	149.6	3.07	1 710 000	2200	2250
Average									1974

TABLE 3.—Summary of Tests of Square-End Struts—Continued

Strut	Length	Breadth	$\frac{L}{r}$	Depth	$\frac{L}{r}$	Aera	modulus of elasticity	Proportional limit	Ultimate unit load
	Inches	Inches		Inches		Inches	Lbs./in.	Lbs./in.	Lbs./in²
D-1	88.375	1.733	176.6	1.743	175.5	3.02	940 000	900	1109
E-2	88.375	1.765	175.5	1.742	175.7	3.07	1 600 000	1300	1539
B-4	88.375	1.741	175.8	1.762	173.7	3.07	1 740 000	1200	1750
C-5	88.375	1.729	177.1	1.758	174.1	3.04	1 870 000	1550	1891
D-6	88.375	1.739	176.0	1.746	175.3	3.04	1 360 000	1100	1530
Average									1564
C-1	101.000	1.741	201.0	1.731	202.1	3.01	1 100 000	600	900
D-2	101.000	1.737	201.4	1.747	200.3	3.03	1 600 000	700	1155
E-3	101.000	1.749	200.0	1.760	198.7	3.08	1 600 000	1150	1299
A-4	101.000	1.742	200.8	1.757	199.1	3.06	1 730 000	1200	1454
B-5	101.000	1.732	202.0	1.758	199.0	3.05	1 680 000	1200	1230
C-6	101.000	1.742	200.8	1.753	199.6	3.05	1 550 000	1050	1210
Average									1210
B-1	113.625	1.735	226.9	1.748	225.2	3.03	1 240 000	700	891
C-2	113.625	1.730	227.4	1.735	226.9	3.00	1 720 000	800	1100
D-3	113.625	1.752	224.8	1.744	225.7	3.06	1 640 000	700	1013
E-4	113.625	1.742	226.0	1.757	224.0	3.06	1 740 000	900	1003
A-5	113.625	1.741	226.1	1.758	223.8	3.06	1 580 000	800	1029
B-6	113.625	1.747	225.4	1.751	224.8	3.06	1 360 000	900	1000
Average									1006
A-1	126.250	1.742	251.1	1.767	247.6	3.08	1 600 000	850	873
B-2	126.250	1.736	252.0	1.741	251.2	3.02	1 600 000	600	950
C-3	126.250	1.761	248.4	1.737	251.8	3.06	1 520 000	600	915
D-4	126.250	1.744	250.8	1.768	247.4	3.08	1 680 000	500	900
E-5	126.250	1.745	250.7	1.768	247.4	3.08	1 560 000	600	871
Average									902

TABLE 4.—Bending Tests of Spruce

Test piece	Fiber stress S from curve at deflection of 0.2 inch	Depth d	Modulus of elasticity	Modulus of rupture
	Lbs./in.²	Inches	Lbs./in.²	Lbs./in.²
B–1	5900	1.743	1 625 000	8600
D–2	5770	1.769	1 544 000	8000
A–3	5600	1.765	1 523 000	6300
C–4	5900	1.750	1 625 000	8600
C–5	5400	1.732	1 496 000	7600
E–6	4720	1.778	1 274 000	7200

TABLE 5.—Compression Test of Round-End Strut E–9

[Tested Aug. 24, 1917, by J. E. B. and L. J. L. Length, 6 feet 3¾ inches; breadth, 1.742 inches; depth, 1.752 inches. Slenderness ratio, 150. Tested on large Emery machine. Compression measured with 30-inch Howard gages. Counterweight of 9 pounds at each end of gages. Deflection measurements taken with Johnson dials.]

Zero correction at end of test, 30 pounds. Ultimate load 3050—30=3020 pounds=990 pounds per square inch.

Total load, pounds	Unit load, pounds per square inch	Compression in 30 inches, inches		Average unit compression, inches per inch	Deflection, inches	
		North gage	South gage		Down	North
305	100	0.0000	0.0000	0.000000	0.000	0.000
610	200	.0011	.0011	.000037	.000	.000
915	300	.0020	.0027	.000078	.001	.000
1220	400	.0034	.0041	.000125	.004	.002
1525	500	.0049	.0050	.000165	.009	.003
1677	550	.0054	.0058	.000187	.014	.004
1830	600	.0061	.0063	.000207	.017	.004
2060	675	.0071	.0069	.000233	.022	.004
2135	700	.0074	.0073	.000245	.024	.004
2287	750	.0080	.0078	.000263	.032	.004
2440	800	.0089	.0088	.000295	.039	.003
2593	850	.0100	.0096	.000327	.051	.002
2745	900	.0111	.0103	.000357	.079	.001
2898	950	.0120	.0110	.000383	.103	.011
3050	1000	.0140	.0131	.000452	.312	.015
3050	1000	.0156	.0147	.000505	.482	.029
3300[a]	....	....	....	....	.537	.039
2980[b]	974	....	....	....	.612	.039
3025	992	....	....	....	.862	.069
3015	989	....	....	....	1.142	....

[a] Removed gages. Load stood at 3300 pounds total.

[b] Removed counterweights. Load stood at 2980 pounds.

TABLE 6.—Summary of Tests of Round-End Struts

Strut	Length	Breadth	Depth	$\frac{L}{r}$	Area	Modulus of elasticity	Proportional limit	Ultimate unit load
	Inches	Inches	Inches		Inches²	Lbs./in.²	Lbs./in.²	Lbs./in.²
C-7	12.625	1.774	1.748	25.0	3.10	1 690 000	2400	4226
D-8	12.625	1.765	1.748	25.0	3.08	1 950 000	2500	4314
E-9	12.625	1.741	1.745	25.0	3.04	3 300 000	2400	5487
Average								4676
C-7	25.250	1.783	1.751	50.0	3.12	1 560 000	2000	3630
D-8	25.250	1.760	1.750	50.0	3.08	1 750 000	2800	3610
E-9	25.250	1.746	1.752	50.0	3.06	2 100 000	2800	4908
Average								4049
C-7	37.875	1.768	1.752	74.9	3.10	1 820 000	2400	2816
D-8	37.875	1.754	1.754	74.8	3.08	1 600 000	2600	2607
E-9	37.875	1.749	1.755	74.8	3.07	2 430 000	3800	3905
Average								3109
A-7	50.500	1.807	1.750	100.0	3.16	1 680 000	1200	1503
B-8	50.500	1.752	1.748	100.1	3.06	1 920 000	1800	1938
C-9	50.500	1.738	1.747	100.1	3.04	2 200 000	1400	2031
Average								1824
B-7	63.125	1.788	1.750	125.0	3.13	1 560 000	800	1065
C-8	63.125	1.756	1.752	124.8	3.08	1 640 000	1000	1117
D-9	63.125	1.742	1.750	125.0	3.05	2 240 000	1400	1437
Average								1206
C-7	75.750	1.786	1.756	149.4	3.13	1 760 000	770	772
D-8	75.750	1.773	1.755	149.5	3.11	1 380 000	500	592
E-9	75.750	1.742	1.752	149.8	3.05	2 380 000	800	990
Average								785
B-7	88.375	1.795	1.753	174.7	3.15	1 880 000	550	640
C-8	88.375	1.768	1.755	174.4	3.10	1 440 000	460	503
D-9	88.375	1.731	1.752	174.8	3.03	2 520 000	680	743
Average								629
A-7	101.000	1.708	1.752	199.7	2.99	1 600 000	360	441
B-8	101.000	1.769	1.750	200.0	3.09	1 460 000	320	354
C-9	101.000	1.730	1.750	200.0	3.03	2 100 000	360	521
Average								439

TABLE 7.—Computations for Determining the Value of k for Use in Plotting the Curve of Fig. 14

$\frac{4P}{EC}$	$\sqrt{\frac{4P}{EC}-1}=-\tan\alpha$	α deg	$\log_{10}\alpha$ deg	$\log_{10} 2\alpha$ radians	$\log_{10} \tan\alpha$	$\log_{10} \log_e k$	$\log_{10} \log_{10} k$	$\log_{10} k$
2	1	135.00	2.13033	0.67324	0.00000	0.67324	0.31102	2.04650
5	2	116.57	2.06659	.60950	.30103	.30847	$\bar{1}$.94625	.88359
10	3	108.43	2.03515	.57806	.47712	.10094	$\bar{1}$.73872	.54793
26	5	101.31	2.00565	.54856	.69897	$\bar{1}$.84959	$\bar{1}$.48737	.30717
101	10	95.71	1.98096	.52387	1.00000	$\bar{1}$.52387	$\bar{1}$.16165	.14510
626	25	92.29	1.96515	.50806	1.39794	$\bar{1}$.11012	$\bar{2}$.74790	.05596
10001	100	90.57	1.95700	.49990	2.00000	$\bar{2}$.49990	$\bar{2}$.13769	.01373

$\log\frac{1}{k}$	$\frac{1}{k}$	$\left(1-\frac{1}{k}\right)$	$\log\left(1-\frac{1}{k}\right)^2$	$\log\frac{4P}{EC}$	$\log\frac{4P}{EC}\left(1-\frac{1}{k}\right)^2$	Log strength ratio	Strength ratio	$\sqrt{\frac{1}{k}}$
$\bar{3}$.95350	0.0090	0.9910	$\bar{1}$.99215	0.30103	0.29318	$\bar{1}$.29888	0.1990	0.0948
$\bar{1}$.11641	.1307	.8693	$\bar{1}$.87834	.69897	.57731	$\bar{1}$.58301	.3828	.3616
$\bar{1}$.45207	.2832	.7168	$\bar{1}$.71080	1.00000	.71080	$\bar{1}$.71650	.5206	.5322
$\bar{1}$.69283	.4930	.5070	$\bar{1}$.41002	1.41497	.82499	$\bar{1}$.83069	.6772	.7021
$\bar{1}$.85490	.7160	.2840	$\bar{2}$.90664	2.00432	.91096	$\bar{1}$.91666	.8254	.8462
$\bar{1}$.94404	.8791	.1209	$\bar{2}$.16485	2.79657	.96142	$\bar{1}$.96712	.9271	.9376
$\bar{1}$.98627	.9689	.0311	$\bar{4}$.98552	4.00004	.98556	$\bar{1}$.99126	.9801	.9843

TABLE 8.—Ratios of Moment of Inertia and Strength Ratios

$\sqrt{\frac{1}{k}}$	Ratio of I at $\frac{L}{3}$ to I_2	Ratio of I at $\frac{5L}{16}$ to I_2	Ratio of stiffness of tapered beam to that of uniform beam	Strength ratio
0.2	0.46	0.41	0.40	0.27
.3	.48	.44	.44	.34
.4	.52	.47	.51	.41
.5	.56	.52	.58	.49
.6	.62	.58	.65	.58
.7	.69	.65	.73	.67
.8	.77	.75	.81	.77
.9	.88	.86	.90	.88

WASHINGTON, April 19, 1919.